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Abstract. The Lie algebra gl(n, C) admits a basis which is a generalization of the Pauli spin operators, as described by Patera and Zassenhaus. In this paper a new expression for the set of generators of the enveloping algebra of gl(n, C) is deduced and new quasi-triangular Hopf algebras with solutions to the quantum Yang–Baxter equation for gl(n, C) are derived. Solutions in eight-vertex form are also given. Properties of the algebras suggest applications to physical theories.

1. Introduction

This work relates a new [1] decomposition of the simple Lie algebras A_n to Hopf algebras and the Yang–Baxter equation [2].

The endeavour to understand models in statistical mechanics [3] by means of their algebraic structure involves quantum groups. Drinfeld [4] and Jimbo [5] have developed a formalism to relate to any simple Lie algebra g a Hopf algebra which is essentially their quantum group, giving a solution R to the quantum Yang–Baxter equation in the form [2]

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{1.1}$$

Explicit in their construction is the Cartan, or root space, decomposition of the Lie algebras, and they postulate an expansion for their quantum operator R of the form

$$R=I+\hbar r+\mathrm{O}(\hbar^2) \tag{1.2}$$

where I is the identity and r is the g$\otimes$g-valued operator corresponding to the classical r-matrix [6]. An R satisfying (1.2) is called quasiclassical. The $\hbar$ above is treated formally as an indeterminate.

The work presented here provides a different framework from that of Drinfeld's to investigate solutions to the Yang–Baxter equation (1.1). This framework has a number of interesting properties. The Lie algebras are presented in a decomposition recently developed by Patera and Zassenhaus [1]. It is not a root space decomposition; instead, the generators are in the form of true Pauli spin operators and certain new generalizations to higher rank A_n. The representation space decomposes into dual disjoint subspaces appropriate to the particle–antiparticle description. We derive a quasi-triangular Hopf algebra based on the Patera–Zassenhaus presentation of the algebras A_n. Here the resulting R-matrix is not quasi-classical. The Hopf algebras are related to the non-standard, or twisted, braid group representations. The first of these non-standard representations was derived by Couture and Lee [7, 8]. Since then, these representations and especially the puzzle of their algebraic structure have received

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wide attention, see, for example, [9, 10]. The theory developed here incorporates a parameter which takes discrete values at the nth roots of unity. Operators in this decomposition are also especially suited to eight-vertex models.

The structure of this paper is as follows. Section 2 summarizes the results of Patera and Zassenhaus to introduce their presentation of the Lie algebras A_n in terms of generalized Pauli generators, and demonstrates our result, crucial to the Hopf algebra applications, that a set of two Pauli generators is sufficient to generate the enveloping algebra of A_n. Section 3 states the Hopf algebra axioms for reference. Section 4 gives a Hopf algebra for A_2 based on the Patera–Zassenhaus decomposition and discusses some of its unusual properties. Section 5 gives a Hopf algebra and solution to the Yang–Baxter equation for arbitrary A_n, shows uniqueness, and discusses limits and deformations. Section 6 gives solutions in the eight-vertex form.

2. The Pauli decomposition

Patera and Zassenhaus [1] have described the properties of the Lie algebras A_n in terms of generalized Pauli spin operators. This is not the decomposition represented by the Gell-Mann matrices.

To be specific, consider the algebra A_2, whose generators span $gl(3, C)$. The eight generators act as follows

$$\begin{array}{lll} D|p\rangle = \xi^p|p\rangle & D^-|p\rangle = \xi^{-p}|p\rangle & \xi = e^{2\pi i/6} \\ A|p\rangle = |p-2\rangle & A^-|p\rangle = |p+2\rangle & \omega = e^{2\pi i/3} \\ B|p\rangle = \xi^p|p-2\rangle & B^-|p\rangle = \xi^{-p}|p+2\rangle & \\ C|p\rangle = \xi^{-p}|p-2\rangle & C^-|p\rangle = \xi^p|p+2\rangle & \end{array} \tag{2.1}$$

The eigenvectors of D are the basis vectors of two disjoint three-dimensional representation spaces

$$p = 0, 2, 4 = \text{even integer mod } 6$$

and

$$p = 1, 3, 5 = \text{odd integer mod } 6.$$

These are interpreted as particle and antiparticle representations. Thus, 3×3 matrices represent the generators. The matrices and the operators they represent have several properties analogous to the Pauli matrices, the 2×2 spin operators. They never annihilate a state, like the step operators of the Cartan decomposition, but they rotate states instead. The adjoint action of any two generators X, Y is of the form

$$XYX^{-1} = \omega^j Y \tag{2.2}$$

for some integer j, for example

$$\begin{aligned} ADA^{-1} &= \omega D \\ DAD^{-1} &= \omega^{-1} A. \end{aligned} \tag{2.3}$$

This decomposition allows, furthermore, a finest grading of the algebra. It suffices to use two non-commuting generators, A and D are chosen, to express all eight. Each generator is labelled uniquely by the eigenvalues a and d of the adjoint action of A and D,

$$\begin{aligned} &X=\omega^k D^d A^a \\ &DXD^{-1}=\omega^d X \\ &AXA^{-1}=\omega^a X. \end{aligned} \tag{2.4}$$

Equations (2.4) generalize to all A_n. Take A and D to be elements of A_{N-1}, and $\omega=\mathrm{e}^{2\pi\mathrm{i}/N}$, and define $D^{-1}=D^{2N-1}$, $D^{-m}=(D^{-1})^m$. From these definitions and equations (2.4) of Patera and Zassenhaus [1] we can assert a remarkable result: A and D generate the enveloping algebra of $\mathrm{gl}(N, C)$. This is in contrast to the case for the Cartan decomposition, where all generators of the Lie algebra are needed to generate the enveloping algebra and it is crucial for Hopf algebras based on the Patera-Zassenhaus decomposition.

3. Hopf algebra axioms

Recall the axioms [4, 11] which define a Hopf algebra A_h. Let A_h be a unital, associative, algebra with multiplication m. It is a Hopf algebra when equipped with the mappings

Δ: co-multiplication, a homomorphism; $A_h \to A_h \otimes A_h$

ε: co-unit, a homomorphism; $A_h \to C$

γ: antipode, an antihomomorphism $A_h \to A_h$

which satisfy, for all a, b in A_h

$$(\mathrm{id}\otimes\Delta)\Delta(a)=(\Delta\otimes\mathrm{id})\Delta(a) \tag{3.1}$$

$$m(\mathrm{id}\otimes\gamma)\Delta(a)=m(\gamma\otimes\mathrm{id})\Delta(a)=\varepsilon(a)\cdot 1 \tag{3.2}$$

$$(\varepsilon\otimes\mathrm{id})\Delta(a)=(\mathrm{id}\otimes\varepsilon)\Delta(a)=a \tag{3.3}$$

Define σ: permutation mapping $A_h\otimes A_h \to A_h \to A_h\otimes A_h$: $\sigma(x\otimes y)=y\otimes x$. There may be an invertible element R in $A_h\otimes A_h$ which satisfies

$$\sigma\cdot\Delta(a)=R\Delta(a)R^{-1}. \tag{3.4}$$

Take the usual convention: for $R=\Sigma a_i\otimes b_i$, R_{kl} means that, for (k, l, m) any permutation of $(1, 2, 3)$, the a_i act in the kth space, the b_i act in the lth space, and the identity in the mth space. If R satisfies

$$\begin{aligned} &(\mathrm{id}\otimes\Delta)R=R_{13}R_{12} \\ &(\Delta\otimes\mathrm{id})R=R_{13}R_{23} \\ &(\gamma\otimes\mathrm{id})R=R^{-1} \end{aligned} \tag{3.5}$$

then the pair (A_h, R) is called a quasi-triangular Hopf algebra. This is sufficient for R to satisfy the Yang–Baxter equation (1.1).

4. Hopf algebra associated with the Pauli operators of gl(3, C)

Consider the enveloping algebra $\tilde{A}$ of A_2 generated by A and D. One may show that $\tilde{A}$ itself is a quasi-triangular Hopf algebra under its multiplication m and the co-multiplication

$$\begin{aligned} \Delta A &= D\otimes A + A\otimes 1 \\ \Delta D &= D\otimes D. \end{aligned} \tag{4.1}$$

For any element a in $\tilde{A}$ equation (3.1) holds, so the co-multiplication (4.1) is co-associative. The co-unit satisfying (3.3) is given by

$$\begin{aligned} \varepsilon(I) &= \varepsilon(D) = 1 \\ \varepsilon(A) &= 0. \end{aligned} \tag{4.2}$$

The antipode satisfying (3.2) is

$$\begin{aligned} \gamma(D) &= D^{-1} \\ \gamma(A) &= -D^{-1}A. \end{aligned} \tag{4.3}$$

So $\tilde{A}$ is demonstrated to be a Hopf algebra under the co-multiplication given in (4.1). This result is independent of representation.

Define the elements e_i of the enveloping algebra with the projection property

$$\begin{aligned} e_0 &= (I + D + D^2)/3 \qquad\qquad e_p|q\rangle = \delta_{pq}|p\rangle \\ e_2 &= (I + \omega^2 D + \omega D^2)/3 \\ e_4 &= (I + \omega D + \omega^2 D^2)/3. \end{aligned} \tag{4.4}$$

Since Δ and ε are homomorphisms, it follows that

$$\begin{aligned} \Delta e_0 &= e_0\otimes e_0 + e_2\otimes e_4 + e_4\otimes e_2 \\ \Delta e_2 &= e_0\otimes e_2 + e_2\otimes e_0 + e_4\otimes e_4 \\ \Delta e_4 &= e_0\otimes e_4 + e_2\otimes e_2 + e_4\otimes e_0 \\ \varepsilon(e_i) &= \delta_{i0}. \end{aligned} \tag{4.5}$$

Since γ is an antihomomorphism it follows that

$$\begin{aligned} \gamma(e_0) &= e_0 \\ \gamma(e_2) &= e_4 \\ \gamma(e_4) &= e_2. \end{aligned} \tag{4.6}$$

An invertible element R which satisfies (3.4) and (3.5) is

$$R = I\otimes e_0 + D^2\otimes e_2 + D\otimes e_4 \tag{4.7}$$

with

$$R^{-1} = I\otimes e_0 + D\otimes e_2 + D^2\otimes e_4 \tag{4.8}$$

where

$$(\mathrm{id}\otimes\gamma)R = (\gamma\otimes\mathrm{id})R = R^{-1}. \tag{4.9}$$

Thus under the mappings (4.1), (4.4) and (4.5), $(\tilde{A}, R)$ is a quasi-triangular Hopf algebra and R is a solution to the Yang–Baxter equation (1.1).

Several properties are of note here. Perhaps most striking mathematically is the form (4.1) of the co-multiplication. In the Drinfeld quantum groups the co-multiplication is co-commutative before the deformation parameter is introduced. The D in the expression for ΔA means that, although no deformation parameter has been introduced, the co-multiplication (4.1) is non-co-commutative. This distinguishes our Hopf algebra from that of Drinfeld and Jimbo. It causes phase shifts in the R matrix, which is diagonal in the basis space $p=0$, 2, 4, as follows

$$\begin{array}{ccccccccc} (0,0) & (0,2) & (2,0) & (0,4) & (2,2) & (4,0) & (2,4) & (4,2) & (4,4) \\ 1 & 1 & 1 & 1 & \omega^2 & 1 & \omega & \omega & \omega^2 \end{array}. \tag{4.10}$$

By contrast, for the standard R-matrix, as $\hbar \to 0$, $R \to I$; all matrix elements equal one on the diagonal.

The D in the co-multiplication for A also implies that the co-unit for any power of D must be one to satisfy the property (3.3), in contrast to $\varepsilon(H)=0$ for standard quantum groups. This is similar to the multiple mapping properties of a quotient algebra, such as the Clifford algebra which describes intrinsic spin.

The expressions (4.1) for the co-multiplication preserve the Pauli grading by A, giving a weight conservation. In the sense that the operator A rotates states the co-product conserves these internal rotations. Preservation of the A grading in physical applications can imply selection rules. The antipode corresponds to the opposite rotation. As seen from the projection form (4.6): the state zero is its own image under the antipode and states two and four result from rotations in opposite directions from state zero. From (5.4), D to any power times its antipode is unity. So when D^m operates on any state $|2k\rangle$, that state is multiplied by ω^{km}, and when $\gamma(D^m)$ operates, the phase factor corresponds to the opposite rotation in the complex plane. The antipode of A is $-A$ times a phase depending on the state p it operates on, so that A and its antipode generate related infinitessimal rotations in the Lie group generated by the algebra.

5. Generalization to A_n

The structures given for A_2 generalize to A_n. Taking A and D as generators of A_n, or, equivalently, of $\mathrm{gl}(N, C)$, $N=n+1$, the commutation rules are (2.3) and the co-multiplication is (4.1). The co-unit and antipode of A and D are just as given by (4.2) and (4.3). These expressions in the Hopf algebra are indepenent of representation. They depend on n only through the definition of ω.

Define the projectors

$$e_{2m}=\sum \omega^{N-jm} D^j/N \tag{5.1}$$

with the projection property as in (4.4). When acting on a state $|2m\rangle$, each term gives one, when acting on any other, they sum to zero. All sums are from 0 to n. Using the convention $e_{2N}=e_0$, it follows that

$$\Delta e_{2m}=\sum e_{2i}\otimes e_{2m-2i} \tag{5.2}$$

$$\varepsilon(e_i)=\delta_{i0}. \tag{5.3}$$

The antipode satisfying (3.2) is

$$\gamma(D^m)=D^{-m}$$

$$\gamma(e_{2m})=e_{2N-2m}. \tag{5.4}$$

An invertible element R which satisfies (3.4) and (3.5) is

$$R = \sum D^{N-m} \otimes e_{2m}. \tag{5.5}$$

The R-matrix is diagonal with elements given by

$$R(2k, 2l; 2k, 2l) = \omega^{-kl}. \tag{5.6}$$

Its inverse is

$$R^{-1} = \sum D^{m} \otimes e_{2m} \tag{5.7}$$

which satisfies both relations in (4.9). It is also represented by a diagonal matrix, and its elements are the inverse, and also the complex conjugate, of the elements of the R-matrix in (5.6).

The expression for R is unique to within an overall constant, given the rules for commutation (2.3) and requiring that R be diagonal in states $|k, l\rangle$, k and l even, and quasi-triangular under the co-multiplication (4.3). To see this, note that R must be expandable in terms of the form $D^{p} \otimes D^{q}$ or, equivalently, of the form $D^{p(m)} \otimes e_{2m}$. The coefficients and the value of p may be determined to give (5.5) by using the commutation rules and

$$Ae_{2m} - e_{2m-2}A = 0. \tag{5.8}$$

The solution (5.5) for A_1 is the same as one of the non-standard braid group representations [7, 8] in the undeformed limit. The limit as $\hbar \to 0$ of the $N = 2$ representation gives (upon the usual transposition) a diagonal R-matrix with elements

$$1, 1, 1, \omega \qquad \text{for } \omega = \exp(2\pi \mathrm{i}/2) = -1. \tag{5.9}$$

Our equation (5.5) implies the same R-matrix.

In our derivations we have not introduced a deformation parameter $\hbar$. However, we wish to argue that the algebra of section 5 is, in some sense, deformed; and that ω, which is given for any gl(N) by [1] $\omega = \exp(2\pi\mathrm{i}/N)$, may be regarded as a deformation parameter. As is well known, R-matrices from quantum groups may be interpreted as scattering matrices. The Drinfeld–Jimbo limit $R = I$ as $\hbar \to 0$ means that in the scattering, states are unchanged. A scattering interpretation of (4.9) is quite different. The phase shift is some ω^{m}, depending on the initial state. As $N \to \infty$, $\omega \to 1$, and the eigenvalues of D, as least in the neighbourhood of some state $|p\rangle$, approach a constant value. Choosing the axes appropriately, $p = 0$, $D \to 1$. The co-product then becomes co-commutative locally, as it should be for an undeformed quantum group. Locally the large R-matrix also approaches a constant, as required for a quasi-classical R-matrix.

That the braiding matrix associated with algebras in the Patera–Zassenhaus grading has a parameter which takes values of the roots of unity suggests relevance to conformal field theory. See also [12] and [13]. After completing this work, [15] and [16] came to our attention.

6. Eight-vertex models

The Patera–Zassenhaus decomposition is especially applicable for models where weights are conserved modulo two, as in the eight-vertex model [3]. This is because the operator A changes the weight p by two units, and m by one. Setting the operator

$B = iAD$ as the third Pauli operator for sl(2), with λ_i arbitrary, the following are solutions to the Yang–Baxter equation (1.1):

$$\begin{aligned} R_1 &= \lambda_2 A \otimes A + \lambda_3 B \otimes B \\ R_2 &= \lambda_3 B \otimes B + \lambda_1 D \otimes D \\ R_3 &= \lambda_1 D \otimes D + \lambda_2 A \otimes A. \end{aligned} \tag{6.1}$$

This may be verified by substituting directly into (1.1) and using the commutation relations for Pauli matrices.

The Sklyanin algebra [14] is expressed in Pauli operators equivalent to A, B, and D, but solutions (6.1) are not equivalent to the eight-vertex solution by R J Baxter which appears in that algebra.

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